AXONOMETRICAL PROJECTIONS

OF THE MOST IMPORTANT GEOMETRICAL SURFACES.

DRAWINGS IN DESCRIPTIVE GEOMETRY.

Serving at the same time as a Catalogue of Models executed according to the aforesaid Projections

BY

Ferdinand Engel.

WITH IX PLATES.

EDITION FOR AMERICA.

PREFACE

BY

Dr. Joachimsthal,

PROFESSOR IN THE UNIVERSITY OF HALLE.

NEW YORK,

H. GOEBELER, 343 BROADWAY.

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1855.

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PREFACE.

The collection of models by Mr. Engel, and the drawings which he is about to publish as a catalogue of his collection, are so very important for the study of Superior Geometry and Optics that I most willingly yield to the desire of Mr. Engel, and ask in a few words, for these interesting publications, the attention of friends of mathematics and of natural philosophy.

It was the model of the wave surface, (Nr. 1 and 2) the first preparation of which presented the main difficulty. For the purpose of giving some notion of the surface with its two sheets and singular points, it was thought sufficient till now to represent its principal sections by means of wires. Mr. Engel was the first to succeed in modelling in wood the solid included between the two sheets; this model—properly dissected—permitting an exact inspection of the shape of these sheets. The Jury of the London exhibition, first section, class Xth, (physical, chemical, and other instruments) on account of that model—which is a masterpiece in its way*—bestowed upon Mr. Engel the prize-medal. To put the value of that acknowledgment in a proper light I remark, that Sir David Brewster was the chairman of the Jury, and Sir John Herschel among its members.

The models No. 3—12 represent the five principal classes of surfaces of the second order, with their circular sections, right lines and lines of curvature. For the construction of these lines of curvature Mr. Engel, at first, made use only of the projections designed by Monge. In doing this he found—which I make a point of, in order to give a notion of the accuracy of his graphical constructions—that the rectilinear diagonals of any square formed by arcs of lines of curvature are equal to each other. This is quite new, for aught I know, at least in this shape; we may derive it from the well known theorem of Ivory if we add to it the remark of Chasles, that a curved line, perpendicularly intersecting a system of confocal surfaces, meets them in corresponding points. The profit which Mr. Engel has derived from this circumstance will be mentioned in these explanations.

The models Nr. 13—20 represent cones, combinations of hyperbolical paraboloids, &c.; Nr. 21—27 several helicoids and screws; Nr. 28—30 three rectilinear oblique planes (not belonging to the family of surfaces just mentioned); Nr. 31 and 32 are two developable surfaces; Nr. 35—37 refer to the theory of spherical curves and their polar curves, and so forth.

I should exceed the limits of an advertisement if I were to dwell ever so little on the interesting geometrical problems, which were to be settled before the preparation of the models.

A few more words about the drawings: Though originally intended to serve as an extensive catalogue for the collection of models, they are of considerable use, when used in teaching Superior and Descriptive Geometry. Those who are not very well acquainted with the methods of Descriptive Geometry, will understand the drawings quite as well, Mr. Engel's method of projecting being remarkably similar to Perspective.

Some time since, Mr. Engel published two numbers of optical drawings which were most favorably received by Geometers, even in foreign countries. I am quite sure, therefore, that every friend of the higher Mathematics will accept with no little pleasure this new publication of the same author.

F. JOACHIMSTHAL,
PROFESSOR IN THE UNIVERSITY OF HALLE.

* The first model made by Mr. Engel is in the possession of Professor Plücker at Bonn, it is mentioned with great praise [but without telling the artist's name] in the: "Einleitung in die höhere Optik von Beer." A second exemplar is in the physical cabinet of the University of Berlin.

CATALOGUE

OF DRAWINGS OF MODELS FOR THE STUDY OF OPTICS AND THE HIGHER BRANCHES OF GEOMETRY,

with the prices of these models made of wood (W.) or of plaster-composition (P.).

The models are to be had by forwarding the amount and 75 cts. for packing, to H. GOEBELER, 343 Broadway, New York.

Besides these models the undersigned has constantly for sale a large number of models and diagrams, made by himself, intended chiefly for instruction in Descriptive as well as in the higher branches of Analytical Geometry.

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1. Fresnel's Wave Surface of Light in double refracting media, on a wooden stand. The 3 axes a, b, c are to each other as $\sqrt{3} : \sqrt{2} : 1$. $a = 80^{mm}$. Wood $60, Plaster $10.

   This model shews the two sheets of the surface and is decomposable by means of four central sections into two parts in four different ways.

2. Inner part of the Wave Surface convexly represented, of W. $8, P. $3.

2*a*. Fresnel's Wave Surface in which the ratio of the axes is 1,53 : 1,32 : 1. $a = 170^{mm}$. W. $100. P. $20.

2*b*. Inner part of the Wave Surface convexly represented, W. $20. P. $5.

3. Triaxial Ellipsoid, shewing two circular sections, P. $3.

3*a*. The same divided into two parts by a circular central section. W. $10. P. $4.

4. The same with the lines of curvature, P. $10.
5. Hyperboloid of two sheets, shewing two circular sections, on wooden support, P. $8.
6. The same with its lines of curvature, P. $12.
7. Hyperboloid of one sheet, shewing two circular sections and some rectilinear generatrices, P. $8.
8. The same with its lines of curvature, P. $12.
9. Elliptical Paraboloid, shewing two circular sections, P. $4.
10. The same with its lines of curvature, P. $6.
11. Hyperbolical Paraboloid (oblique plane), P. $6.
12. The same with its lines of curvature, P. $8.
13. Right Elliptical Cone, with two circular sections, on wooden support, P. $8.
14. The same with its lines of curvature, P. $10.
15. Double Right Circular Cone, with 3 sections, P. $8.
16. Oblique Circular Cone, with four sections, W. $10.
17. Combination of a Sphere and a Right Elliptical Cone, intersecting each other in two circles, W. $6.
18. Body, enclosed between two squares and four oblique planes, P $5.
19. Body, enclosed between one square and four oblique planes, W. $6.
20. Parallelopiped, divided by an oblique plane into two unequal parts, Zinc $8. P. $6.

    Four Screw-surfaces with their nuts shewing the rectilinear generatrices in different positions.

21. Right Helicoid, P. $6 to 9.
22. Oblique Helicoid, P. $6 to 9.
23. General Helicoid, P. $6 to 9.
24. Developable Helicoid, P. $6 to 9.
25. The same with its lines of curvature, P. $8 to 10.
26. Screw of four grooves, P. $6 to 8.
27. Screw of five grooves, P $6 to 8.
28. Rectilinear oblique surface W. $10 to 12.
29. Another surface of the same kind (elliptical wedge), W. $9. P. $5.
30. A third surface of the same kind, (semi-circular wedge), W. $6. P. $ 3.
31. Developable surface, W. $10. P. $6.
32. Another developable surface, W. $10. P. $6.
33. Serpentine body, P. $6 to 8.
34. Annular body, P. $2.
35. Spherical Curve, with its polar curve, P. $5 to 6.
36. Sphere with four main circles, P. $1 to 2.
37. Spherical Triangle with its symmetrical and polar triangle. W. $8. P. $4.
38. Combination of five Cubes, a crystaline form found sometimes in pyrites, P. $4.
39. Same body, each of the five cubes differently coloured, P. $6 to 8.
40. Four Measuring-scales on wood, intended for the axonometrical method of projection, W. $3.
41. Cube, W. $00.30.
42. Three axis intersecting each other perpendicular, W. $00.30.

For a more detailed description of the surfaces enumerated in this catalogue see the "Explanations."

PRIZE MEDAL OF THE EXHIBITION. F. ENGEL CLASS X.

AXONOMETRICAL PROJECTIONS

OF THE MOST IMPORTANT GEOMETRICAL SURFACES

DRAWINGS OF DESCRIPTIVE GEOMETRY

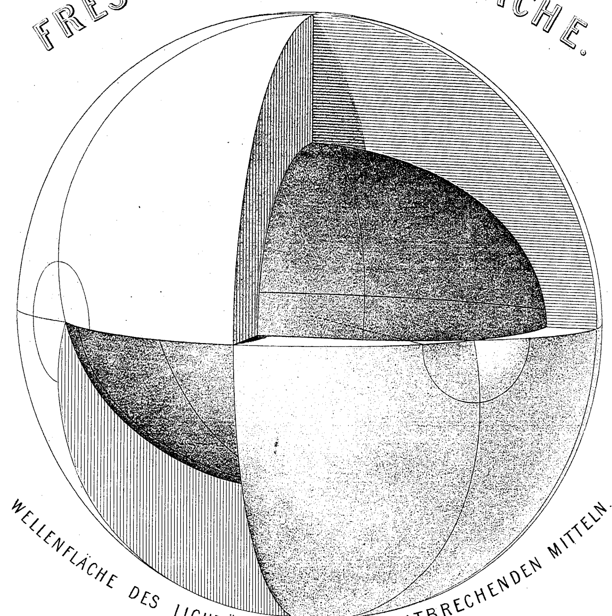
SERVING IN THE SAME TIME AS A CATALOGUE OF MODELS
CARRIED OUT ACCORDING TO THE AFORESAID PROJECTIONS

BY

FERDINAND ENGEL.

WITH IX PLATES.

FRESNEL'S WELLENFLÄCHE.



WELLENFLÄCHE DES LICHTÄTHERS IN DOPPELTBRECHENDEN MITTELN.

AXONOMETRISCHE PROJECTIONEN

DER WICHTIGSTEN GEOMETRISCHEN FLÄCHEN

VORLEGEBLÄTTER FÜR BESCHREIBENDE GEOMETRIE

ZUGLEICH ALS CATALOG EINER MODELLSAMMLUNG VON KÖRPERN
DIE NACH DEN VORGENAÑTEN PROJECTIONEN AUSGEFÜHRT WORDEN SIND

VON

FERDINAND ENGEL.

MIT IX FIGURENTAFELN.

PRIZE MEDAL OF THE EXHIBITION. F. ENGEL CLASS X.

PROJECTIONS AXONOMÉTRIQUES

DES SURFACES GÉOMÉTRIQUES LES PLUS IMPORTANTES

MODÈLES DE DESSIN DE GÉOMÉTRIE DESCRIPTIVE

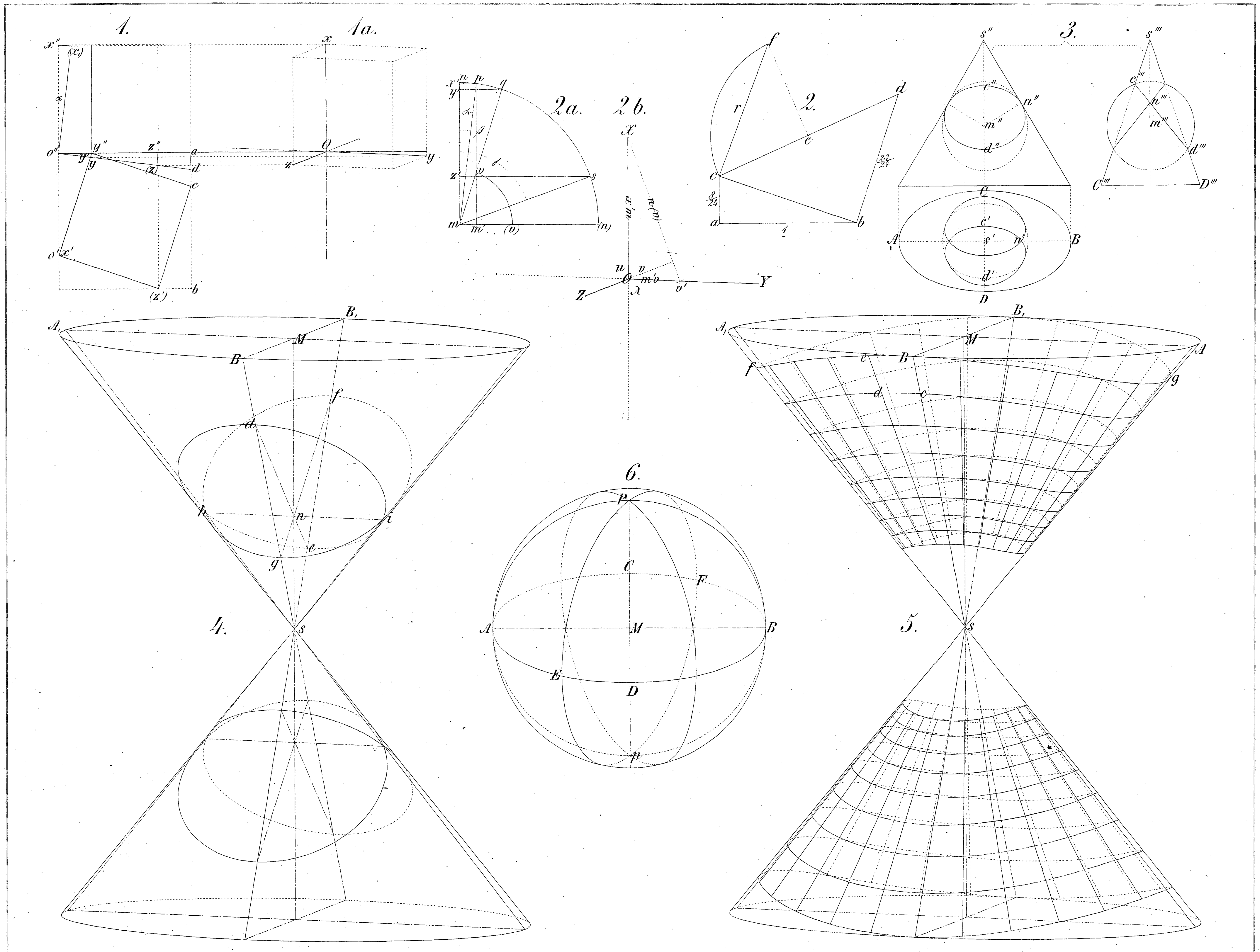
SERVANT EN MÊME TEMPS DE CATALOGUE D'UNE COLLECTION
DE SOLIDES DE GÉOMÉTRIE CONSTRUITS SUR CES PROJECTIONS

PAR

FERDINAND ENGEL.

AVEC IX PLANCHES.

Taf. I.



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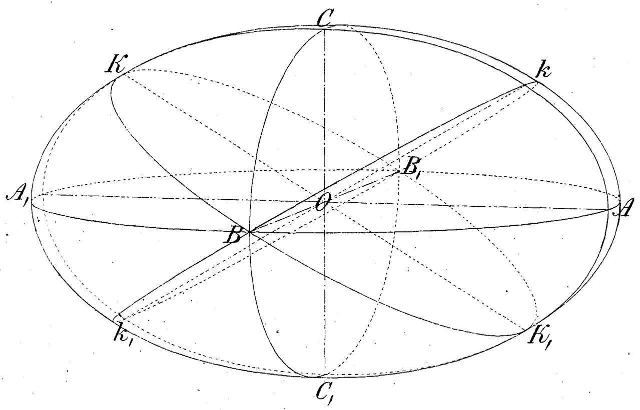

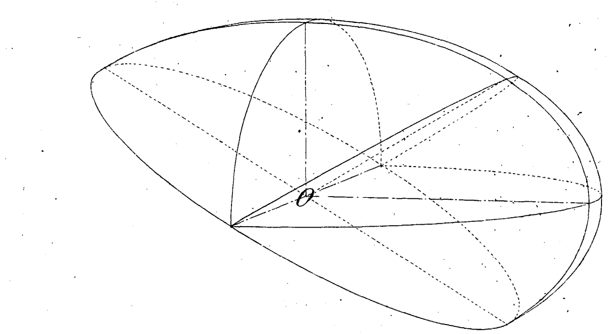

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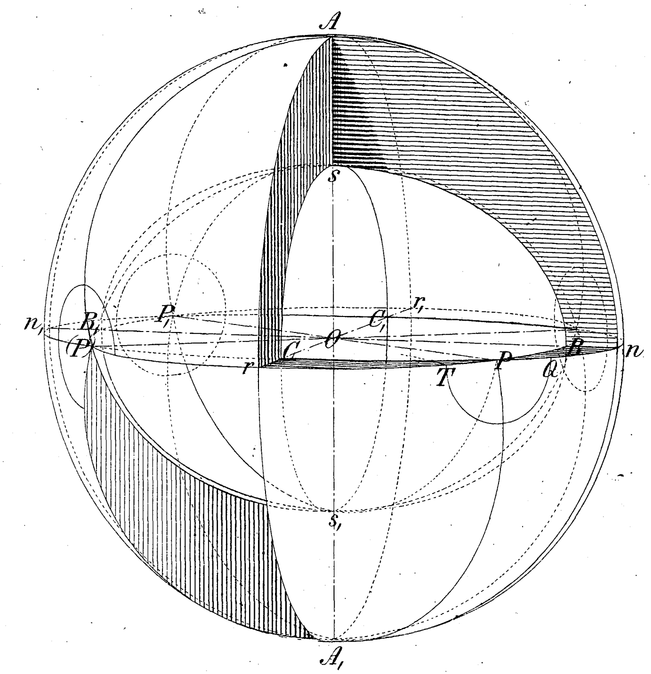

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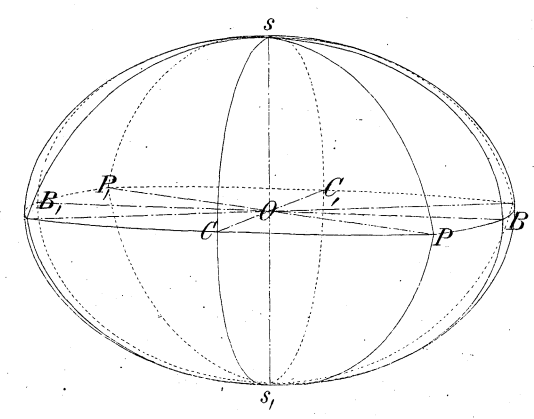

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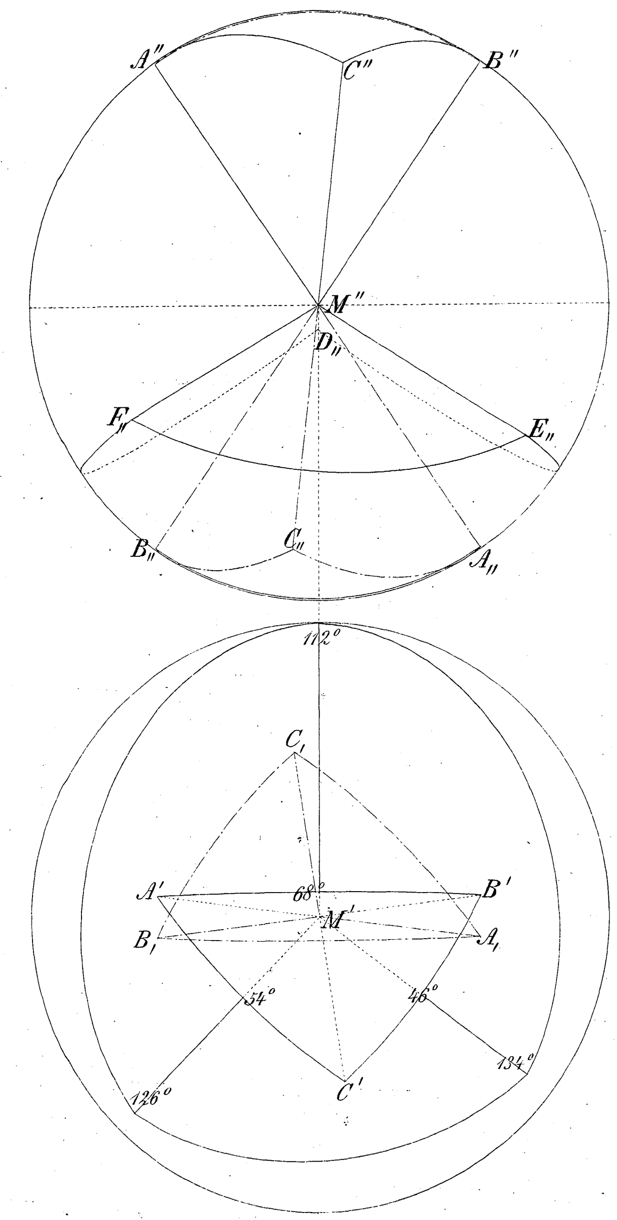

const. u. grav. v. F. Engel.

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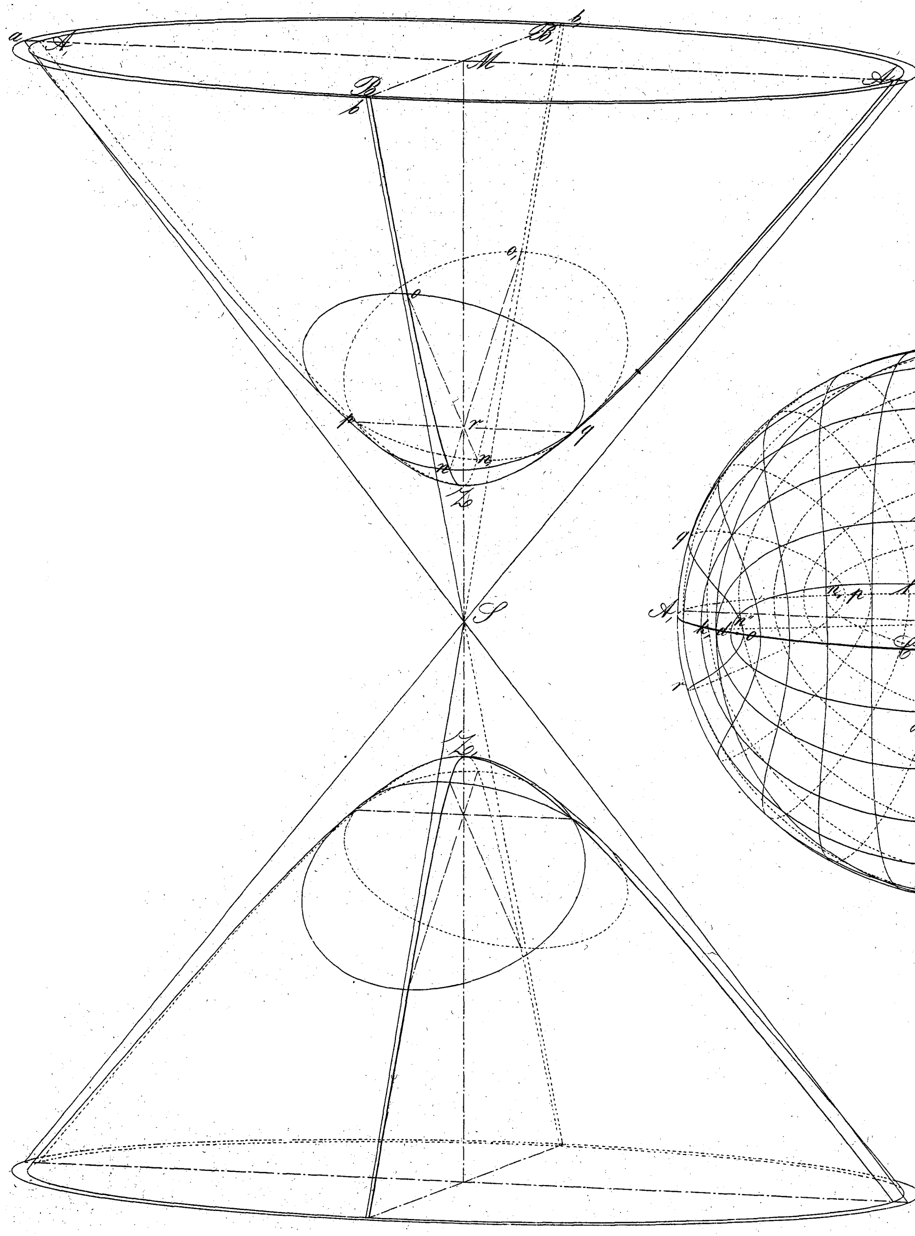

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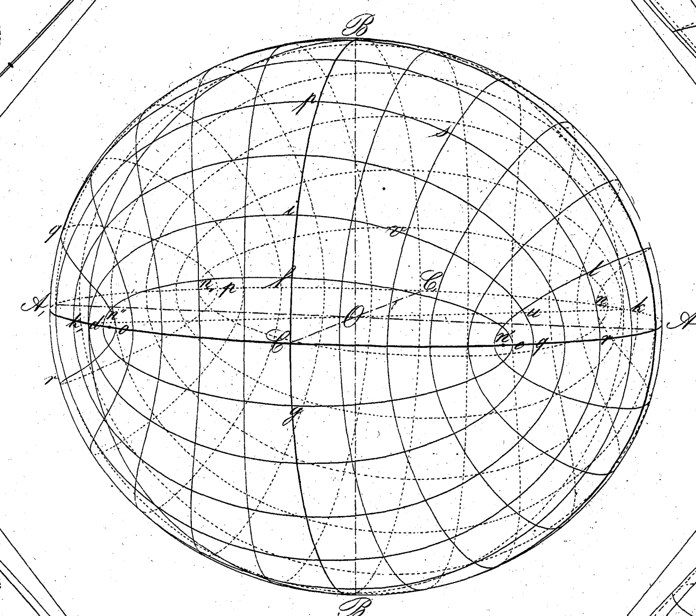

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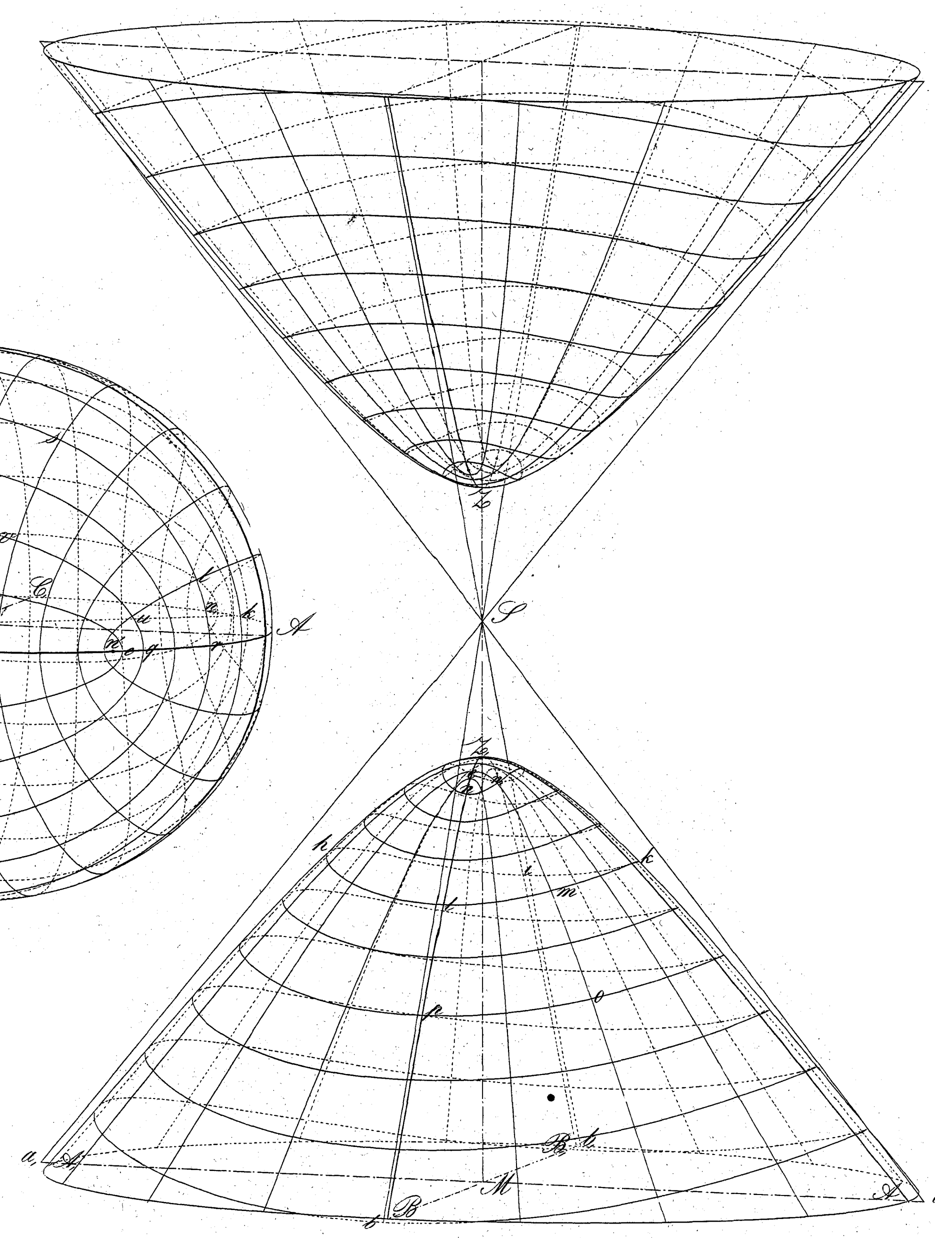

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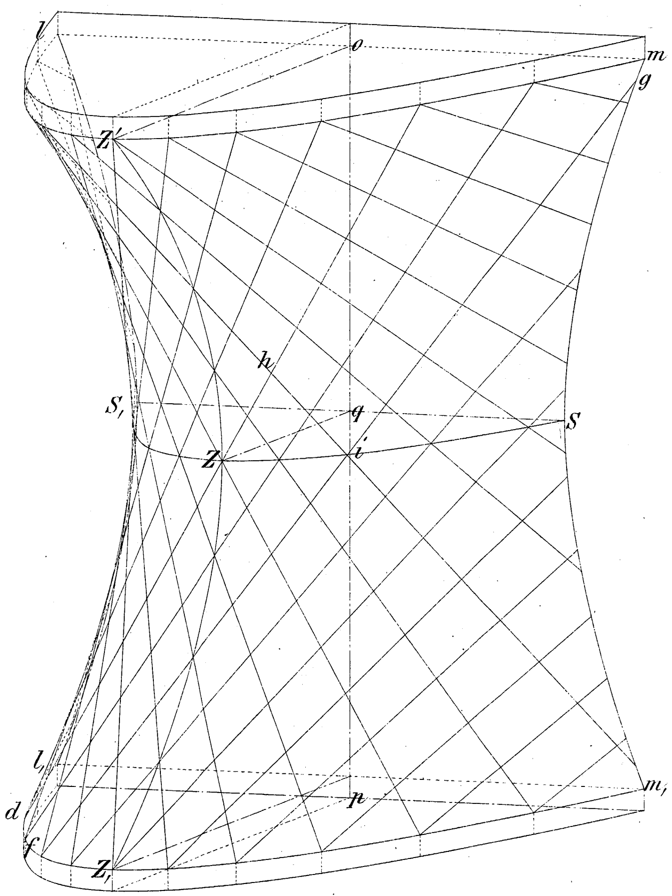

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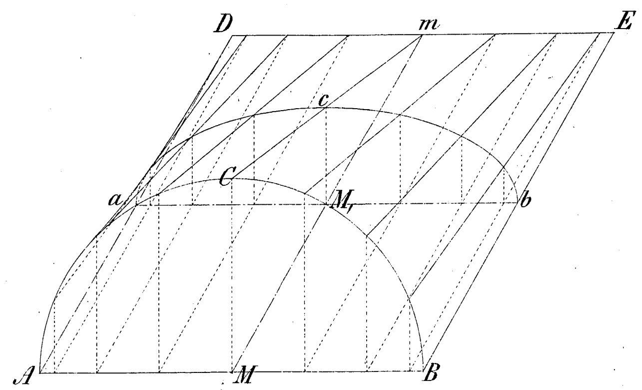

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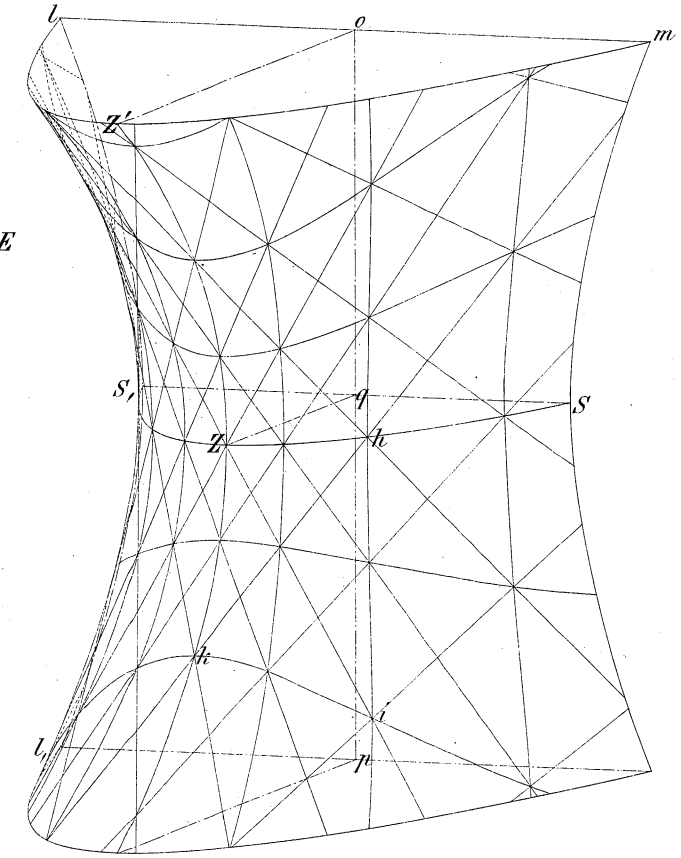

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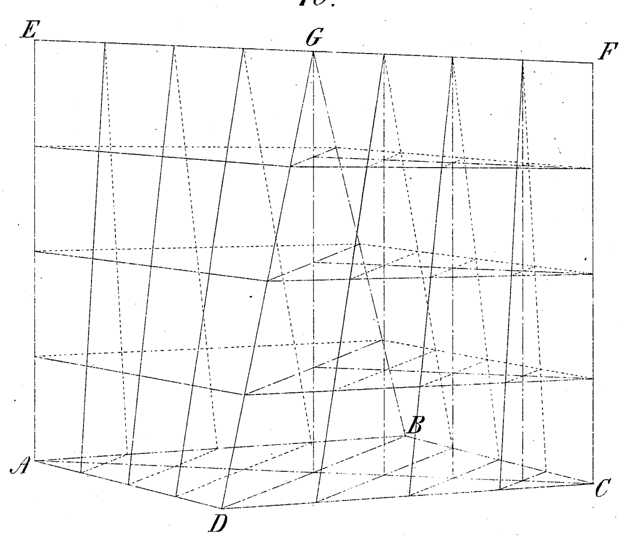

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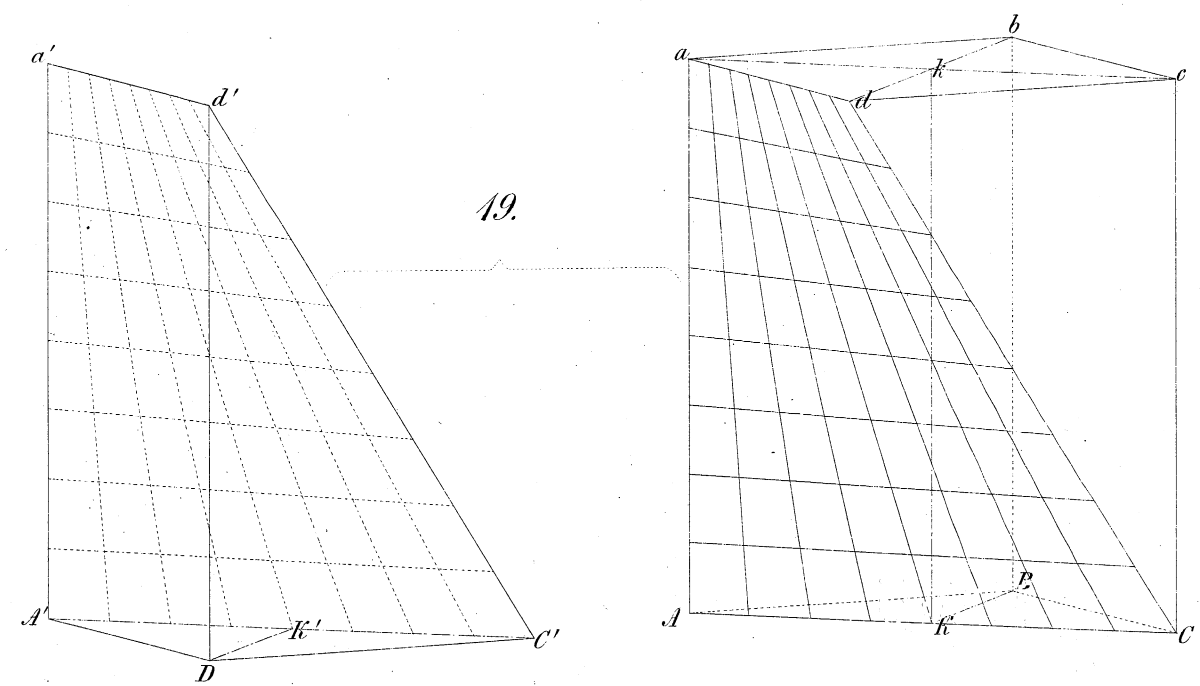

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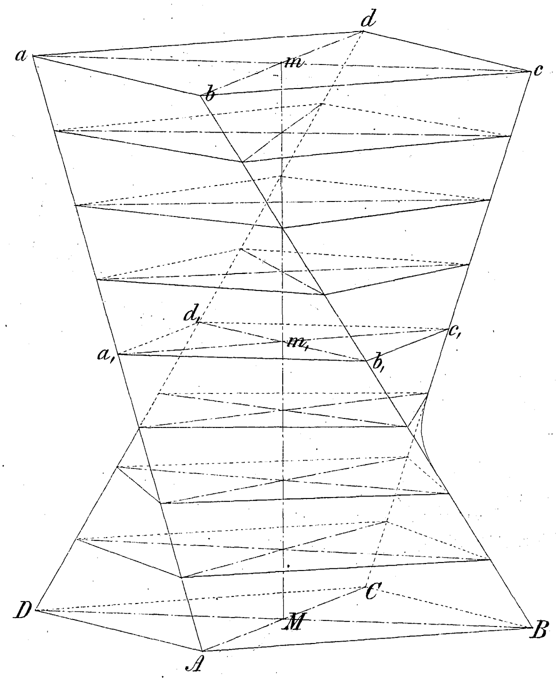

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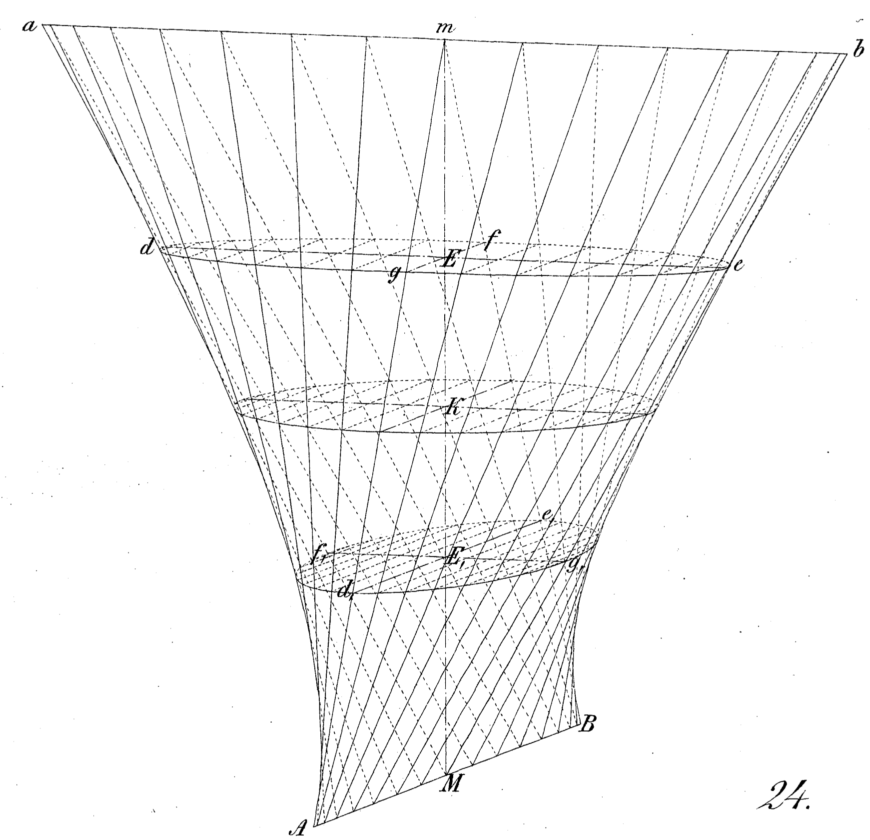

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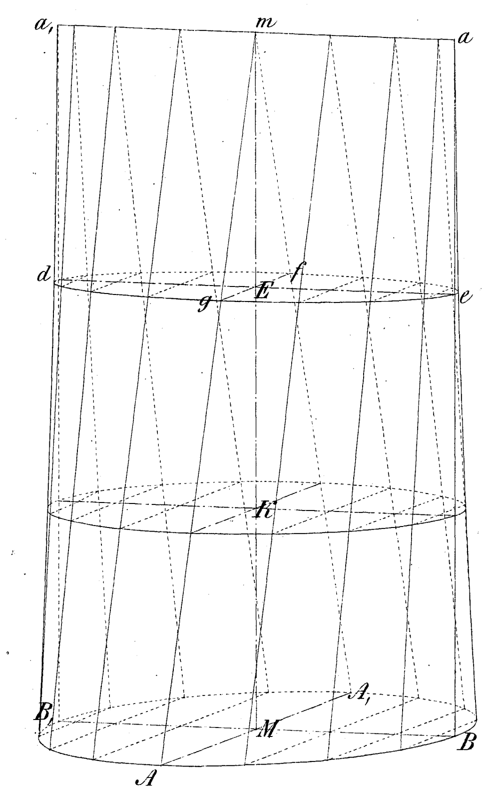

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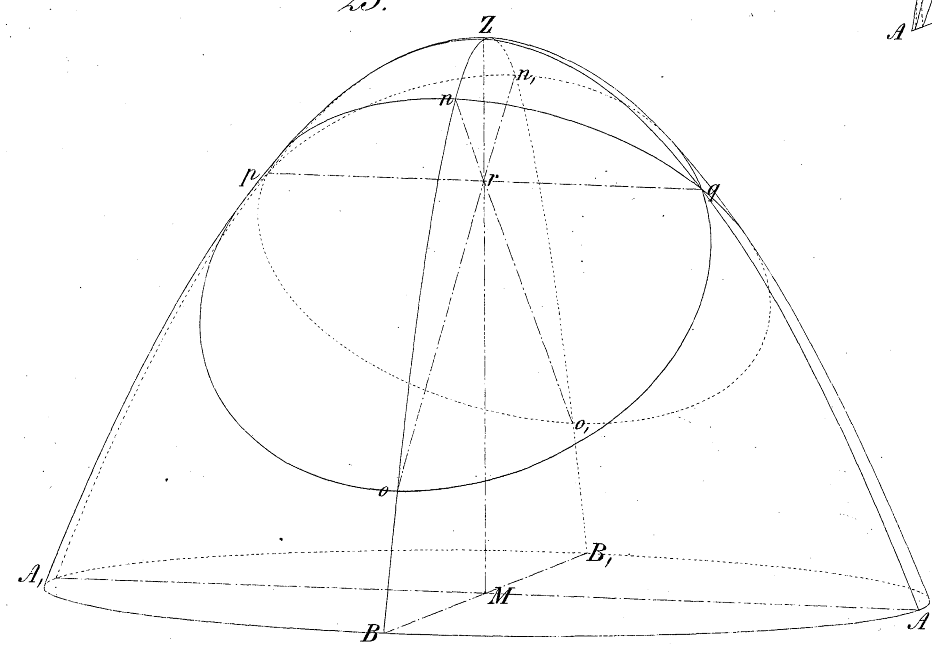

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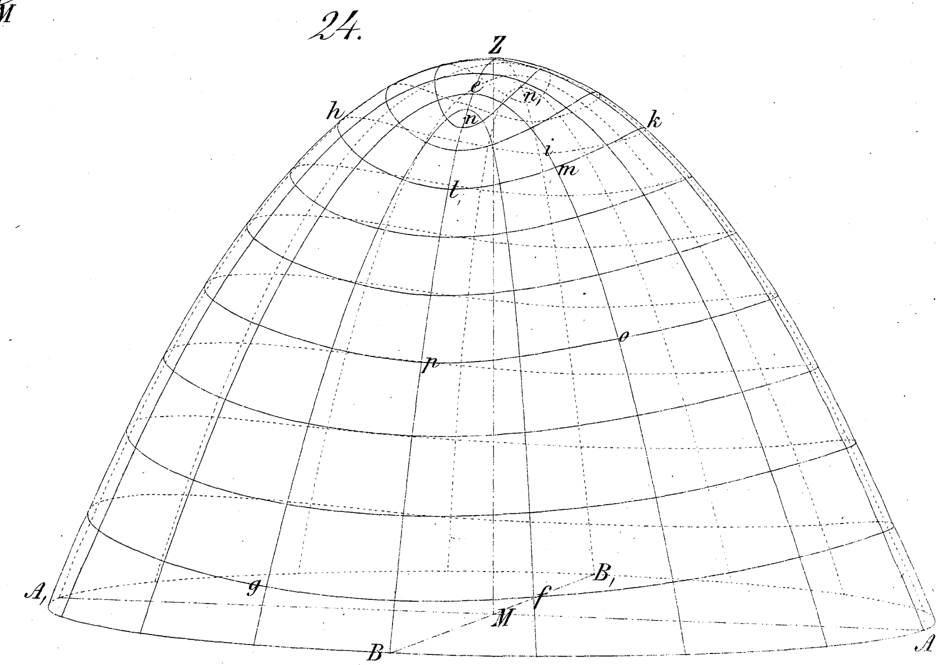

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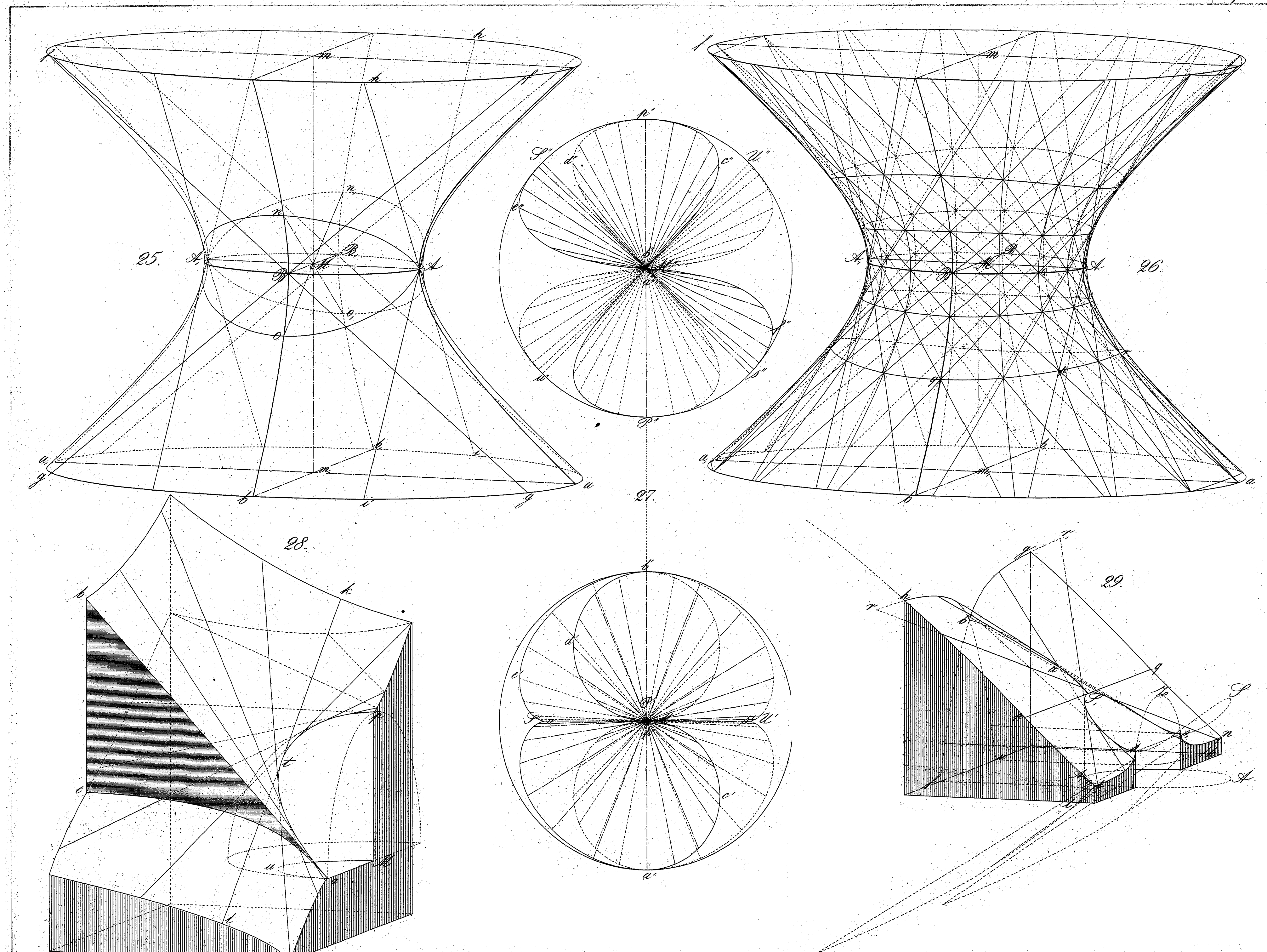

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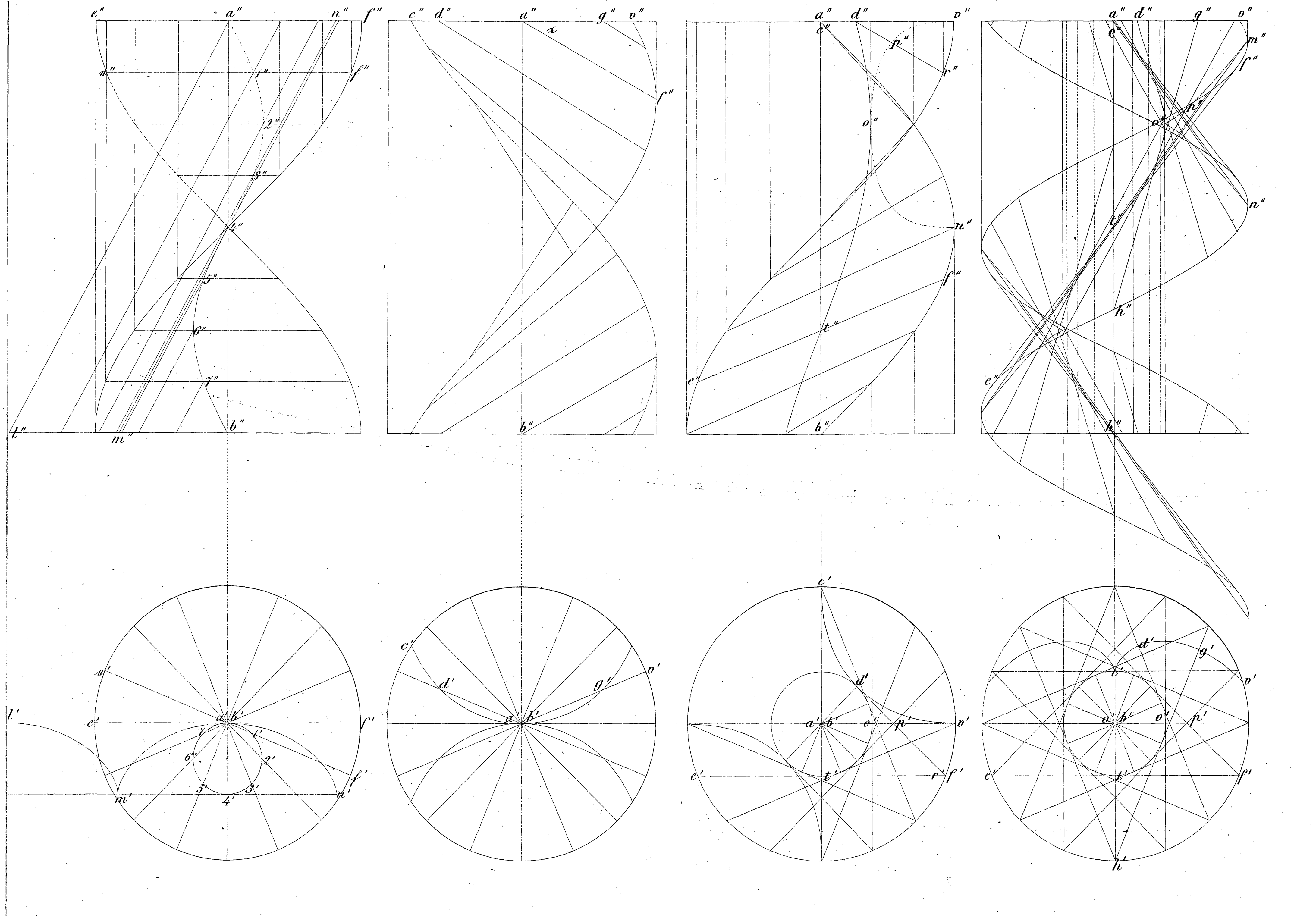
30. 31. 32. 33.



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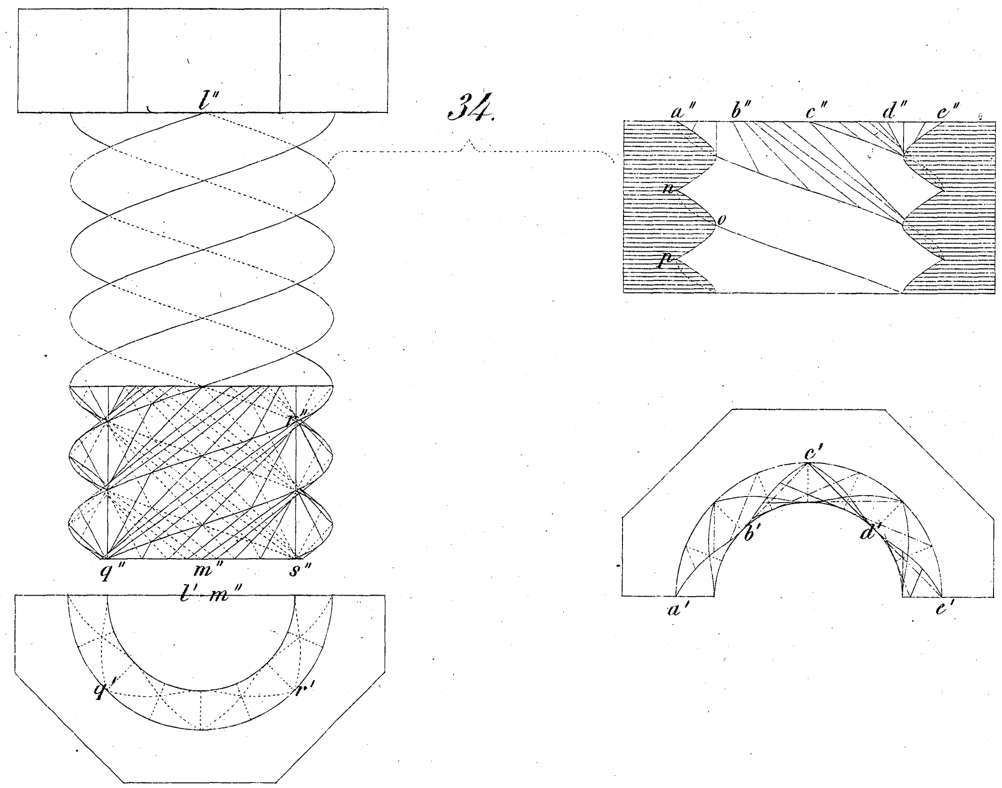

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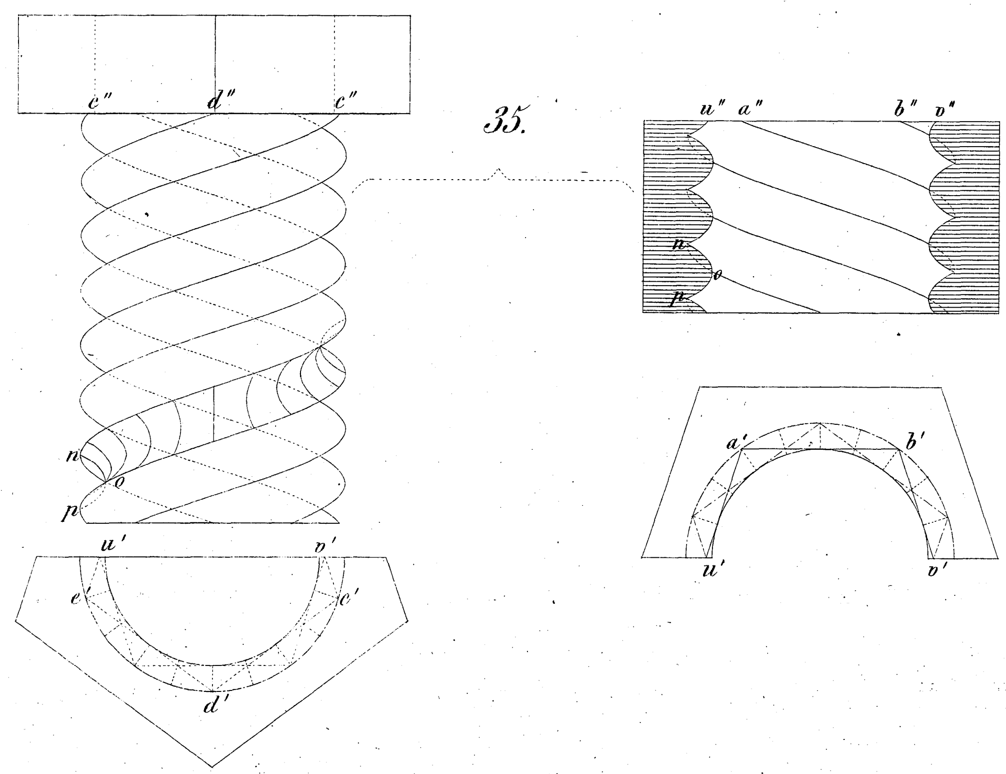

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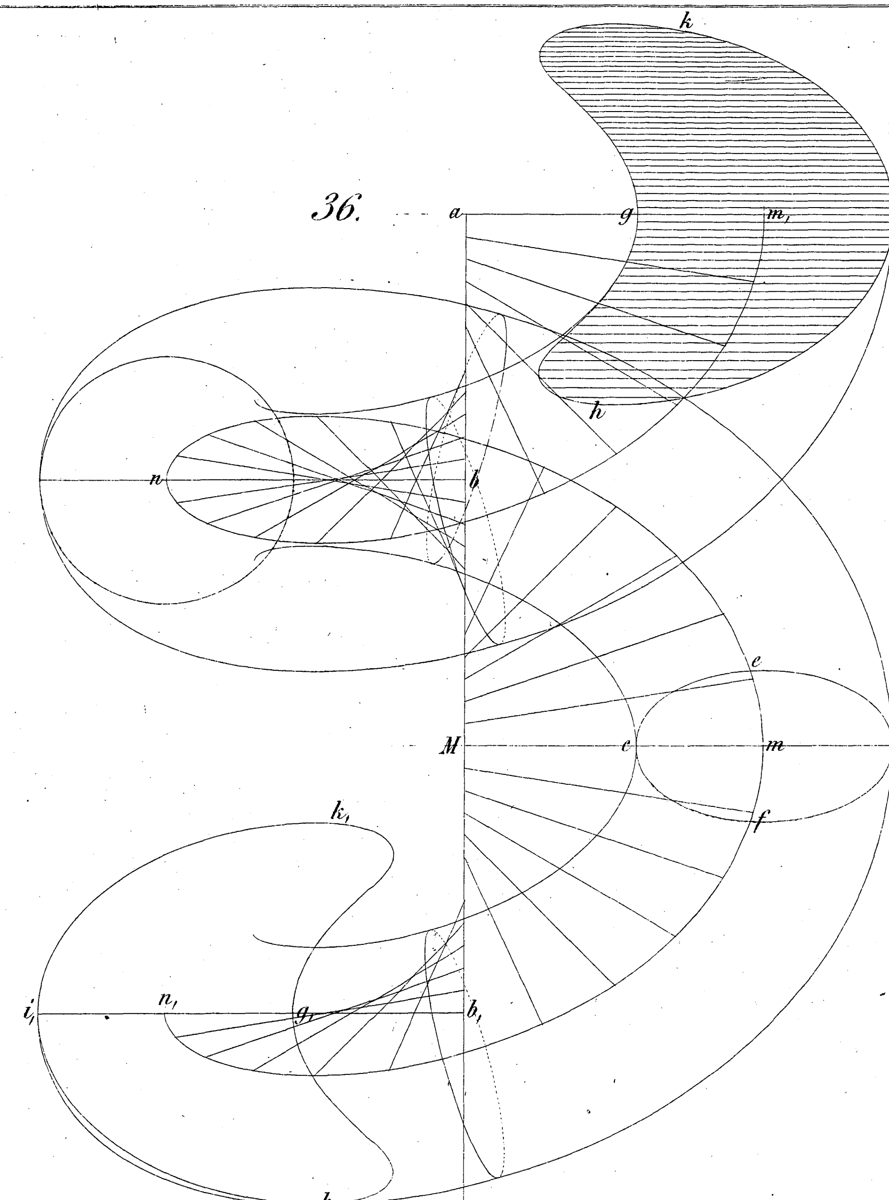

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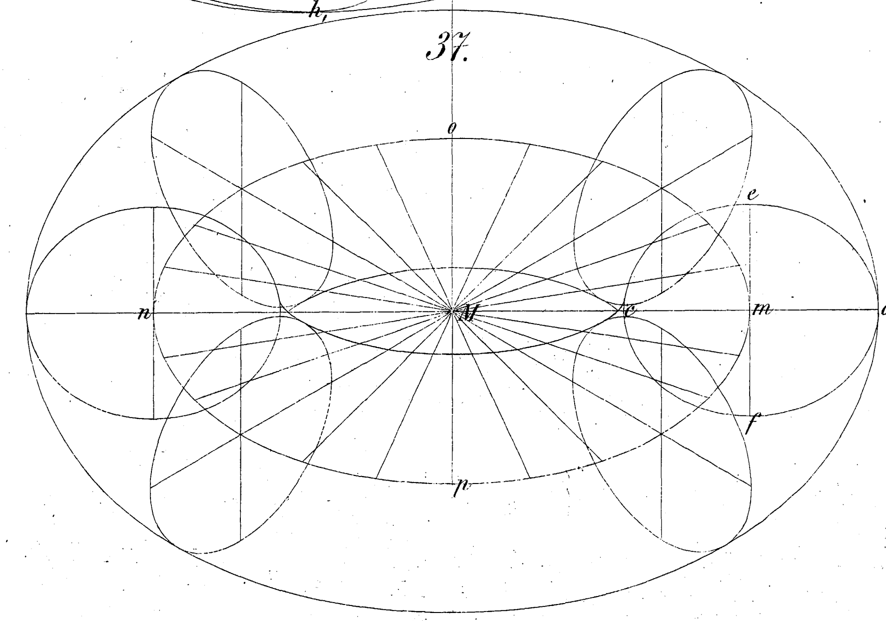

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Taf. IX.

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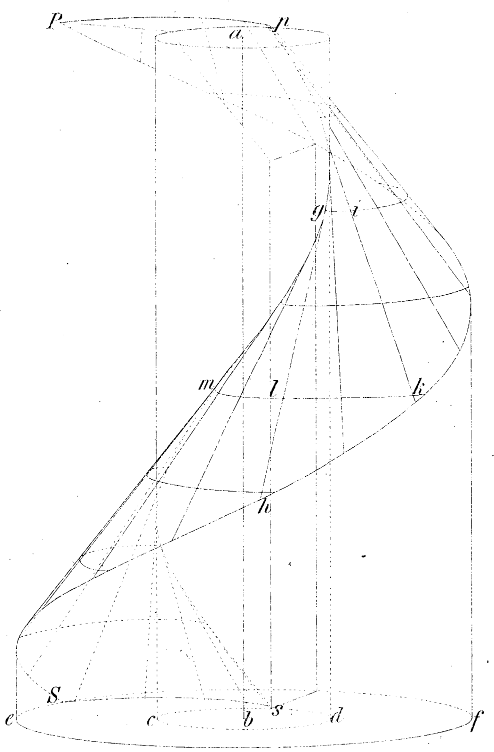

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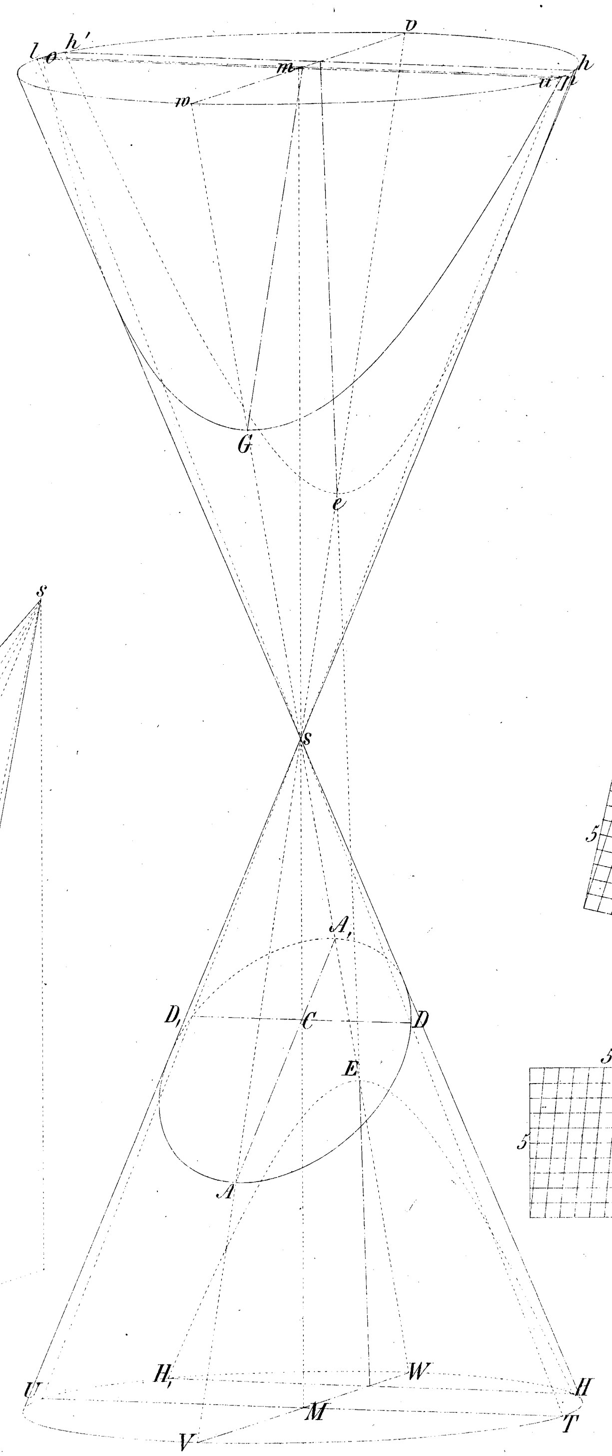

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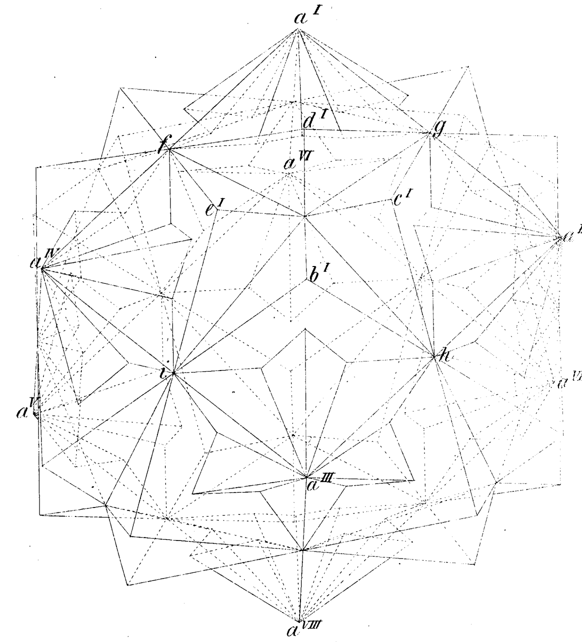

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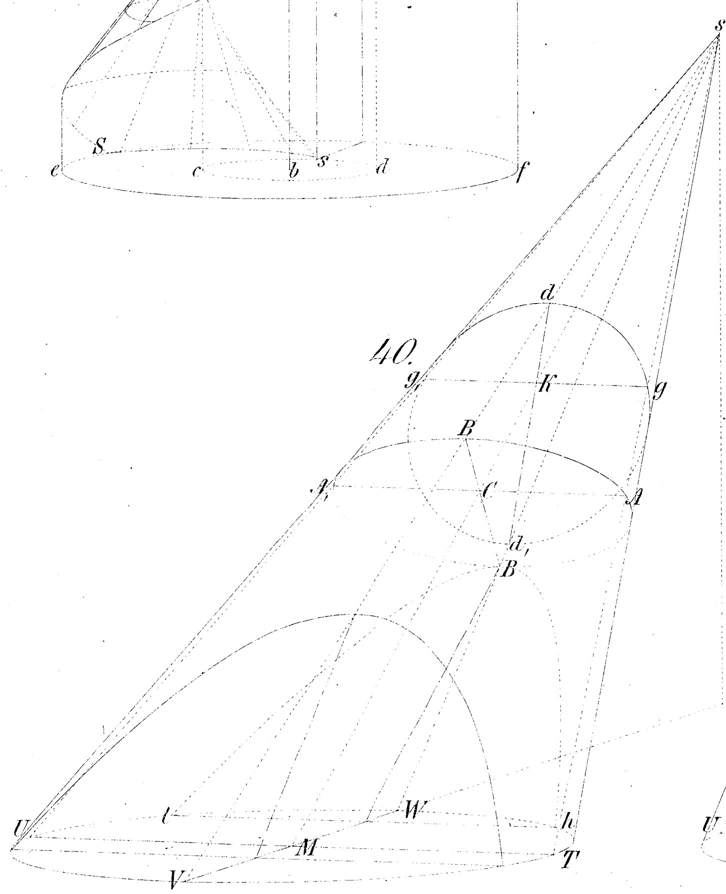

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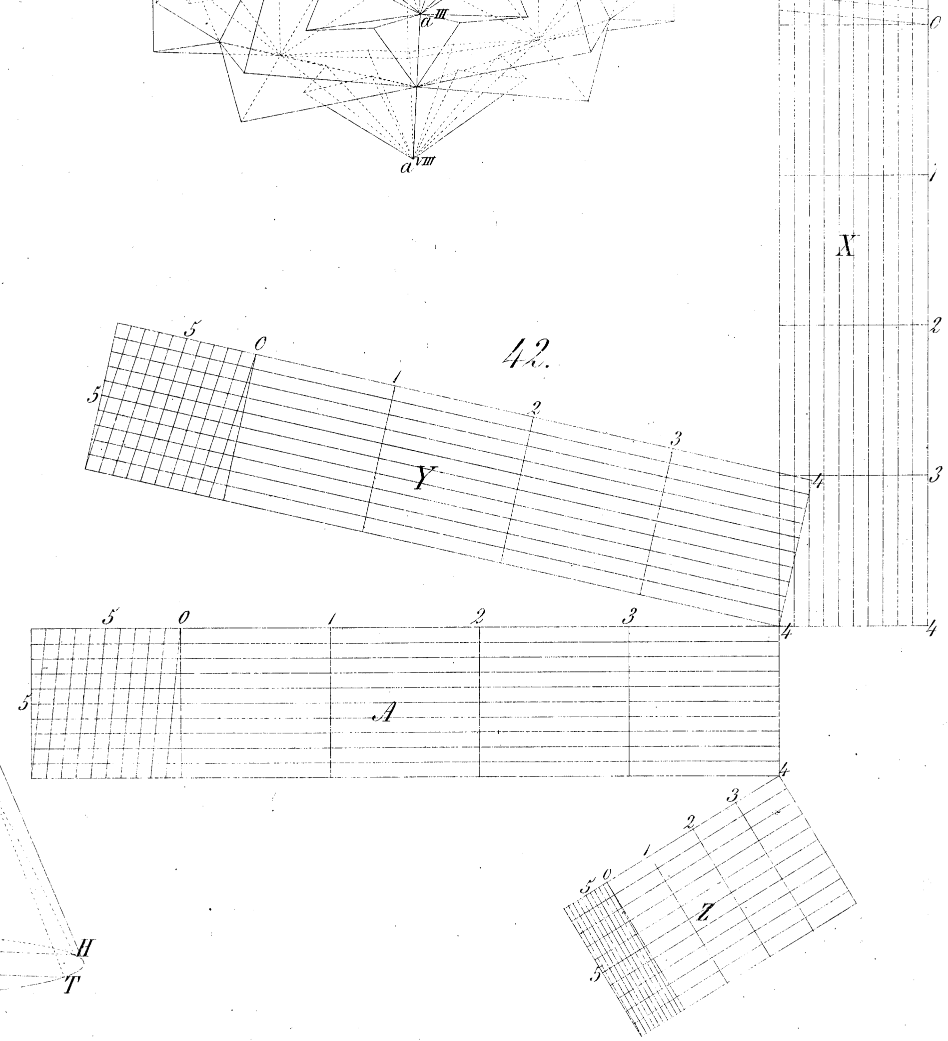

const. u. grav. v. F. Engel.

EXPLANATIONS.

Towards the end of these explanations, some details will be entered into, respecting the particular projection—the axonometrical projection of some German authors—which has been made use of for the drawings.

1. Fresnel's wave surface in biaxial crystals.

(Plate II, fig. 9.)

Draw through the centre O of a triaxial ellipsoid any plane whatever, E, which will intersect the surface in an ellipse; erect in O a perpendicular to E, take on either side of O the lengths of the semi-axes of the ellipse; the resulting four points will be four points of the wave-surface. Giving all possible directions to the transversal plane E, you obtain the complete surface. It consists of two sheets, which correspond to the major and minor axes of the intersection-ellipses; and since among these ellipses there are only two with two equal axes—that is to say: which are circles—the two sheets will only have four points in common.

The wave surface has *three* principal sections, each of which consists of a circle and an ellipse. Suppose, to prove it, a system E_1, of planes drawn through an axis A of the ellipsoid; for all the ellipses, which will result, A will be the principal axis, whereas the second axis will be a diameter of the ellipsoid, perpendicular to A. Construct the points of the wave, which belong to all the planes E_1, and you will see, that they form a circle, the diameter of which is equal to the axis A, and an ellipse, the principal axes of which are equal to the two other axes of the ellipsoid. The circle will be exterior to the ellipse, if A be the greatest axis, it will be interior to the ellipse, if A be the least axis; and if A be the mean axis, the circle will intersect the ellipse at four points.

Fig. 9, Pl. II represents the wave-surface, the upper part on the right hand and the lower one at the left hand are withdrawn, to shew better the interior of the surface. The first principal section consists of a circle $A\,n\,A_1 n_1$ which incloses the ellipse $B\,s\,B_1 s_1$. The second principal section consists of an ellipse $A\,r\,A_1 r_1$ which incloses the circle $s\,C s_1 C_1$. The third principal section consists of an ellipse $n\,C n_1 C_1$, and of a circle $B\,r\,B_1 r_1$, the diameter of which is equal to the mean axis of the ellipsoid. These two curves meet in four points, two of which are denoted by P and P_1. While the tangents drawn through a point of a surface are in general situated in one single plane the tangent-plane, they form for the points P a cone of the second order. The discovery of these singular points of the surface is due to Sir William Hamilton of Dublin. The straight line PP_1, and a second straight line symmetrically situated with respect to the former are called secondary optical axes.—Two sections have been drawn through these axes and through the axis $s\,s_1$, which are indicated in the figure. The points T and Q are the points of contact of one of the four tangents common to the circle and the ellipse, the point T belongs to the circle. The plane drawn through $T\,Q$ perpendicular to the principal section, touches the surface along a circle of which $T\,Q$ is a diameter. The figure shows these four circles of contact, the straight line $O\,T$ is one of the two optical axes.

2. Figure 11 represents the Body inclosed by the interior sheet.

We will add, that PB, $P_1 B_1$ are arcs of a circle, and that PC, $P_1 C_1$ are elliptical arcs.

3. Triaxial Ellipsoid with two circular sections.

(Fig. 7 and 8, Pl. II.)

Fig. 7 represents the triaxial ellipsoid which has been made use of in the construction of the wave-surface: the axes are to each other in the proportion $\sqrt{3} : \sqrt{2} : 1$. Two diameters, KK_1 and kk_1 of the ellipse $A\,C\,A_1 C_1$ are equal to the mean axis BB_1, it follows, that the sections containing the mean axis and one of these two diameters, are circles; the only ones the planes of which pass through the centre of the surface. It is known, that every section parallel to one of these two central sections is a circle.

Fig. 8 represents an ellipsoid divided into two halves by a circular section.

For the better understanding of the following we add these remarks: The name: "Line of curvature of a Surface" is given to a curved line along which the normals of the surface form a developable surface. Every surface has two systems of lines of curvature which intersect each other at right angles. Among all the plane sections drawn through a normal to the surface, those which contain the tangents to the lines of curvature, have the greatest or the least curvature, giving for the concavo-convex surfaces, a different sign to the curvature of the sections, which are in a different position to the tangent plane. The points of the surface, where all the normal sections have the same curvature are called: its *umbilici*. For the theory of the lines of curvature we are indebted to Monge.

4. Ellipsoid with its lines of curvature.

(Fig. 14, Pl. III.)

Figure 14 represents the two systems of the lines of curvature of the ellipsoid; the curve $d\,e\,f\,g$, for instance, belongs to the first system, the curve $o\,q\,p\,r$ to the other. The principal sections $A\,B\,A_1 B_1$ and $B\,C\,B_1 C_1$, drawn through the mean axis, belong to different systems. As for the third section $A\,C\,A_1 C_1$, two of its arcs, $n''n'$ and $n_{11}n_1$, are the limit-curves of the second system. The four points $n'n_1$, $n''n_{11}$ are the umbilici of the ellipsoid. The tangent-planes of the surface in these points are parallel to the two systems of circular sections; the circular section $k_1 B\,k\,B_1$ has been traced.

The lines of curvature of an ellipsoid can be constructed by the help of these four points, for—according to a most remarkable theorem due to Mr. Michael Roberts of Dublin—they bear with respect to the lines of curvature, a part analogous to that of the foci with respect to an ellipse; i. e.: if we fix the extremities of a thread to two umbilici which are not diametrically opposed, and stretch it by means of a style, thus bringing it close to the surface, the style, gliding over the surface, will describe one of its lines of curvature; but this is not the way by which the author determined them. He has found that in every quadrilateral formed by

$L = r \cos a,\ M = r \cos \beta,\ N = r \cos \gamma,$

whence

$L^2 + M^2 + N^2 = r^2\ (\cos a^2 + \cos \beta^2 + \cos \gamma^2)$

but

$\cos a^2 + \cos \beta^2 + \cos \gamma^2 = 2$

thus

$r = \sqrt{\tfrac{1}{2}\,(L^2 + M^2 + N^2)}$

a formula which allows of a very simple construction. We have executed this construction supposing: $L = 1$, $M = \frac{23}{24}$, $N = \frac{8}{24}$ (see fig. 2, pl. I.) These same numbers are adopted too for the scales (fig. 42, pl. IX). The scale designated by A refers to the unity of length, while X, Y, Z are scales of reduction for the three axes. To come back to our problem: Decribe with the radius r a circle about m, take on the radius $m\,n$ the lengths $m\,x' = L$, $m\,y' = M$, $m\,z' = N$, erect the perpendiculars $x'p$, $y'q$, $z's$: then the angles $p\,m\,x'$, $q\,m\,y'$, $s\,m\,z'$ will be equal to the inclinations a, β, γ.

These being found, the problem of determining the angles between the projections of the axes, the units of the scales of reduction being given, is to be solved in the following manner: Suppose through p—*that* extremity of the edge of the cube which is the nearer one to the plane of projection —a plane parallel to the latter. It is obvious, that this new plane will determine on the second axis a segment equal to $m\,v$. The rectangular triangle between the first axis and the segment of the second one, will have for its projection a second triangle one side of which is equal to the known hypotenuse of the triangle in space, while the two other sides are equal to $m\,x'$ and to the projection of $m\,v$ on $m\,n$, i. e. to $m'\,v$.

With the help of these three right lines the triangle $X\,O\,v'$ (fig. 2*b*) has been constructed in which the angle in O is the angle between the projections of the two axes. The position of the projection of the third axis will be found by drawing $O\,Z$ perpendicular to $X\,v'$. Indeed, the third axis being perpendicular to the plane of the projected triangle, its projection must be perpendicular to the trace of this plane; but $X\,v'$ is parallel to this trace; $O\,Z$, therefore, must be perpendicular to Xv'.

The problem we have just solved is a special case of another well known problem, viz of the reduction of an angle to the horizon.

TESTIMONIALS.

I have great pleasure in offering my testimony to the extraordinary beauty and value of Mr. Engel's Collection of Geometrical and Optical Models. It is not always easy for a student to obtain clear and accurate conceptions of the relations of space from diagrams alone, and there are probably few teachers who have not often felt the want of a complete and well executed series of models. Mr. Engel's beautiful collection will be found to be a most desirable addition to the usual method of instruction in Geometry and Optics, and I earnestly recommend it to the attention of teachers and students.

WOLCOTT GIBBS,

Prof. Chemistry and Physics in the Free Academy in New York. June 26th, 1855.

TWO LETTERS TO THE AUTHOR.

Beifolgend erlaube ich mir die mir gütigst mitgetheilten Zeichnungen nebst Text mit besonderem Danke zu remittiren. Ich weiss in der That nicht, ob ich Ihre "schönen Modelle" höher schätzen soll oder "die Zeichnungen," und glaube, dass beide in Verbindung mit einander beim Unterricht erst recht fruchtbar sein werden. Vollenden die erstern die dem Anfänger oft schwierige Vorstellung der Flächen und Linien, um die es sich handelt, so zeigen die letztern die graphische Darstellung in einer Vollkommenheit, die nichts zu wünschen übrig lässt und in vielen Fällen mehr leistet als die körperliche Construction.

Der Text ist zum Verständniss dessen, was gegeben ist, ausreichend. Für den Gebrauch, der in dem Bereich meiner Wirksamkeit von den Blättern gemacht werden wird, hätte ich an manchen Stellen ein Eingehen auf die Entstehungsweise der Figuren, eine Angabe der Constructionen gewünscht. Dadurch hätten die Blätter zugleich als eine Aufgaben-Sammlung gedient. Indessen verkenne ich nicht, dass die Arbeit, wie sie vorliegt, für das grössere Publikum besser passt. Ich wünsche, dass das letztere recht gross sein möge, und kann ich hierzu etwas beitragen, so wird es mir zum Vergnügen gereichen.

Berlin, 14. Juli 1854.

DRUCKENMÜLLER,

Director des Königl. Gewerbeinstitutes.

Mr. Engel de cette ville, duquel la rare perfection avec laquelle il exécute toute sorte de dessins géometriques et la précision vraiment surprenante avec laquelle il construit en relief les objets si variés sur lesquels s'exerce la haute Géometrie, ont valu depuis longtemps dans ce pays-ci la vive approbation de tous les professeurs qui ont été à même d'apprécier les grands secours que les dessins et les modèles de Mr. Engel fournissent à l'enseignement, étant sur le point de s'expatrier, je me plais à exprimer la haute estime que j'ai conçue pour son beau talent. Puisse son talent aussi rare qu'il est utile, avoir aussi ailleurs tout le succès qu'il mérite.

Berlin, le 7 Août 1854.

G. LEJEUNE DIRICHLET,

Membre de l'Académie de Berlin, associé de l'Institute de France.